Linear Algebra 1

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1 Linear equations

$(4+3+3+5+2+4+2+2=25$ pts)

A drone is flying in three-dimensional space. Its unknown position, relative to a chosen reference point, is

$$P=(x,y,z).$$

Recall that if $P=(x,y,z)$ and $Q=(a,b,c)$, then the distance between P and Q is

$$\operatorname{dist}(P,Q)=\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}.$$

Consequently, the *squared distance* between the two points is

$$\operatorname{dist}(P,Q)^2=(x-a)^2+(y-b)^2+(z-c)^2.$$

Four positioning beacons are located at

$$Q_0=(0,0,0),\qquad Q_1=(1,2,1),\qquad Q_2=(-2,1,3),\qquad Q_3=(3,-1,-2).$$

The measured squared distances between the drone and the beacons are

$$\operatorname{dist}(P,Q_0)^2=14,\qquad \operatorname{dist}(P,Q_1)^2=14,\qquad \operatorname{dist}(P,Q_2)^2=20,\qquad \operatorname{dist}(P,Q_3)^2=26.$$

(a) Write the four equations determined by the distance measurements.

(b) The resulting equations are not initially linear. Subtract the equation corresponding to Q_0 from each of the other three equations.

(c) Show that all quadratic terms cancel, and write the resulting linear system in the form

$$A\begin{bmatrix}x\\y\\z\end{bmatrix}=\boldsymbol{b}.$$

(d) Use elementary row operations to put the augmented matrix into row echelon form.

(e) Determine whether the system is consistent. Justify your answer.

(f) Continue using elementary row operations to obtain the reduced row echelon form of the augmented matrix.

(g) Identify the lead and free variables.

(h) Obtain the general solution.

Required Knowledge: **Systems of linear equations, elementary row operations, row echelon form, reduced row echelon form, consistency, and lead/free variables.**

Solution:

1a: The four equations are

$$\begin{aligned}x^2+y^2+z^2&=14,\\(x-1)^2+(y-2)^2+(z-1)^2&=14,\\(x+2)^2+(y-1)^2+(z-3)^2&=20,\\(x-3)^2+(y+1)^2+(z+2)^2&=26.\end{aligned}$$

1b: Subtracting the first equation from the other three gives

$$\begin{aligned} -2x - 4y - 2z + 6 &= 0, \\ 4x - 2y - 6z + 14 &= 6, \\ -6x + 2y + 4z + 14 &= 12. \end{aligned}$$

1c: After simplification, the linear system is

$$\begin{aligned} x + 2y + z &= 3, \\ -2x + y + 3z &= 4, \\ 3x - y - 2z &= 1. \end{aligned}$$

Equivalently,

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & -1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}}_{\boldsymbol{b}}.$$

1d: We apply elementary row operations to the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ -2 & 1 & 3 & 4 \\ 3 & -1 & -2 & 1 \end{array}\right] \xrightarrow[\textcircled{3} \leftarrow \textcircled{3} - 3 \cdot \textcircled{1}]{\textcircled{2} \leftarrow \textcircled{2} + 2 \cdot \textcircled{1}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 10 \\ 0 & -7 & -5 & -8 \end{array}\right].$$

Next,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 5 & 5 & 10 \\ 0 & -7 & -5 & -8 \end{array}\right] \xrightarrow[\textcircled{3} \leftarrow \textcircled{3} + 7 \cdot \textcircled{2}]{\textcircled{2} \leftarrow \frac{1}{5} \cdot \textcircled{2}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 6 \end{array}\right] \xrightarrow{\textcircled{3} \leftarrow \frac{1}{2} \cdot \textcircled{3}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

The last matrix is in row echelon form.

1e: The system is consistent because the row echelon form has no row of the form $\left[\begin{array}{ccc|c} 0 & 0 & 0 & r \end{array}\right]$ with $r \neq 0$.

1f: Continuing from the row echelon form, we obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[\textcircled{1} \leftarrow \textcircled{1} - 1 \cdot \textcircled{3}]{\textcircled{2} \leftarrow \textcircled{2} - 1 \cdot \textcircled{3}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{\textcircled{1} \leftarrow \textcircled{1} - 2 \cdot \textcircled{2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

1g: The variables x, y, z are all lead variables. There are no free variables.

1h: The system has the unique solution

$$\boxed{(x, y, z) = (2, -1, 3)}.$$

Indeed, this point satisfies all four original squared-distance equations.

2 Determinants and nonsingularity (16 + 4 = 20 pts)

Let

$$M = \begin{bmatrix} a+1 & a & a & a \\ b & b+1 & b & b \\ c & c & c+1 & c \\ d & d & d & d+1 \end{bmatrix}.$$

(a) Calculate $\det(M)$ using elementary row and column operations. Clearly indicate how each operation affects the determinant.

(b) Determine the values of a, b, c, d for which M is invertible.

REQUIRED KNOWLEDGE: **Elementary row and column operations, their effects on determinants, triangular matrices, and nonsingularity.**

SOLUTION:

2a: We first apply elementary **column** operations:

$$\det(M) = \begin{vmatrix} a+1 & a & a & a \\ b & b+1 & b & b \\ c & c & c+1 & c \\ d & d & d & d+1 \end{vmatrix} \xrightarrow[\textcircled{4} \leftarrow \textcircled{4} - 1 \cdot \textcircled{1}]{\textcircled{2} \leftarrow \textcircled{2} - 1 \cdot \textcircled{1} \\ \textcircled{3} \leftarrow \textcircled{3} - 1 \cdot \textcircled{1}} \begin{vmatrix} a+1 & -1 & -1 & -1 \\ b & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ d & 0 & 0 & 1 \end{vmatrix}.$$

These column operations do not change the determinant.

Next, we apply elementary **row** operations:

$$\begin{vmatrix} a+1 & -1 & -1 & -1 \\ b & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ d & 0 & 0 & 1 \end{vmatrix} \xrightarrow[\textcircled{1} \leftarrow \textcircled{1} + 1 \cdot \textcircled{4}]{\textcircled{1} \leftarrow \textcircled{1} + 1 \cdot \textcircled{2} \\ \textcircled{1} \leftarrow \textcircled{1} + 1 \cdot \textcircled{3}} \begin{vmatrix} a+b+c+d+1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ d & 0 & 0 & 1 \end{vmatrix}.$$

These row operations also leave the determinant unchanged. The resulting matrix is lower triangular, and hence

$$\boxed{\det(M) = a + b + c + d + 1}.$$

2b: The matrix M is invertible if and only if $\det(M) \neq 0$. Thus,

$$\boxed{M \text{ is invertible if and only if } a + b + c + d \neq -1}.$$

3 Eigenvalues and eigenvectors (7 + 6 + 2 + 5 = 20 pts)

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$. Thus, AB is an $m \times m$ matrix whereas BA is an $n \times n$ matrix.

(a) Suppose that $\lambda \neq 0$ is an eigenvalue of AB with corresponding eigenvector v. Prove that Bv is an eigenvector of BA corresponding to λ.

(b) Prove the converse: if $\lambda \neq 0$ is an eigenvalue of BA, then λ is also an eigenvalue of AB.

(c) Conclude that AB and BA have the same nonzero eigenvalues.

(d) Explain why the condition $\lambda \neq 0$ is necessary.

REQUIRED KNOWLEDGE: **Matrix multiplication, eigenvalues and eigenvectors.**

SOLUTION:

3a: Since $\boldsymbol{v}$ is an eigenvector of AB corresponding to λ,

$$AB\boldsymbol{v} = \lambda\boldsymbol{v}, \qquad \boldsymbol{v} \neq \mathbf{0}.$$

First, $B\boldsymbol{v} \neq \mathbf{0}$. Indeed, if $B\boldsymbol{v} = \mathbf{0}$, then

$$AB\boldsymbol{v} = A(B\boldsymbol{v}) = \mathbf{0}.$$

It would follow that $\lambda\boldsymbol{v} = \mathbf{0}$. Since $\lambda \neq 0$, this would imply $\boldsymbol{v} = \mathbf{0}$, a contradiction. Moreover,

$$BA(B\boldsymbol{v}) = B(AB\boldsymbol{v}) = B(\lambda\boldsymbol{v}) = \lambda(B\boldsymbol{v}).$$

Since $B\boldsymbol{v} \neq \mathbf{0}$, it is an eigenvector of BA corresponding to λ.

3b: Suppose that $\lambda \neq 0$ is an eigenvalue of BA, with eigenvector $\boldsymbol{w}$. Then

$$BA\boldsymbol{w} = \lambda\boldsymbol{w}, \qquad \boldsymbol{w} \neq \mathbf{0}.$$

If $A\boldsymbol{w} = \mathbf{0}$, then

$$BA\boldsymbol{w} = B(A\boldsymbol{w}) = \mathbf{0},$$

contradicting

$$BA\boldsymbol{w} = \lambda\boldsymbol{w} \neq \mathbf{0}.$$

Hence, $A\boldsymbol{w} \neq \mathbf{0}$. Furthermore,

$$AB(A\boldsymbol{w}) = A(BA\boldsymbol{w}) = A(\lambda\boldsymbol{w}) = \lambda(A\boldsymbol{w}).$$

Thus, $A\boldsymbol{w}$ is an eigenvector of AB corresponding to λ.

3c: Parts (a) and (b) prove both inclusions. Therefore,

$$\boxed{AB \text{ and } BA \text{ have the same nonzero eigenvalues}}.$$

3d: The condition $\lambda \neq 0$ is used to guarantee that $B\boldsymbol{v}$ and $A\boldsymbol{w}$ are nonzero. The conclusion can fail for the eigenvalue zero. For example, take

$$A = \begin{bmatrix}1 & 0\end{bmatrix}, \qquad B = \begin{bmatrix}1\\0\end{bmatrix}.$$

Then

$$AB = \begin{bmatrix}1\end{bmatrix}, \qquad BA = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}.$$

Thus, zero is not an eigenvalue of AB, but it is an eigenvalue of BA.

4 Diagonalization (5 + 8 + 7 + 5 = 25 pts)

Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

(a) Determine the characteristic polynomial of A.

(b) Decide whether A is diagonalizable. Justify your answer.

(c) If A is diagonalizable, find a nonsingular matrix X and a diagonal matrix D such that

$$A = XDX^{-1}.$$

(d) Use the diagonalization to calculate A^n for every positive integer n.

REQUIRED KNOWLEDGE: **Characteristic polynomials, eigenvalues and eigenvectors, and diagonalization.**

SOLUTION:

4a: Using the convention $p_A(\lambda) = \det(\lambda I - A)$,

$$\begin{aligned} p_A(\lambda) &= \det \begin{bmatrix} \lambda & -1 & 1 \\ 1 & \lambda - 2 & 1 \\ 1 & -1 & \lambda \end{bmatrix} \\ &= \lambda(\lambda-1)^2 + (\lambda - 1) + (1 - \lambda) \\ &= \boxed{\lambda(\lambda-1)^2}. \end{aligned}$$

Thus, the eigenvalues are 0, with algebraic multiplicity one, and 1, with algebraic multiplicity two.

4b: For $\lambda = 0$, we solve $A\boldsymbol{x} = 0$. Using elementary row operations,

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{\textcircled{1} \leftrightarrow \textcircled{2}} \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{\textcircled{3} \leftarrow \textcircled{3} - 1 \cdot \textcircled{1}} \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

Continuing,

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[\textcircled{1} \leftarrow -1 \cdot \textcircled{1}]{\textcircled{3} \leftarrow \textcircled{3} + 1 \cdot \textcircled{2}} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\textcircled{1} \leftarrow \textcircled{1} + 2 \cdot \textcircled{2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, we see that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is an eigenvector for the zero eigenvalue.

For $\lambda = 1$, we solve $(A - I)\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} \xrightarrow[\substack{\textcircled{1} \leftarrow -1 \cdot \textcircled{1}}]{\substack{\textcircled{2} \leftarrow \textcircled{2} - 1 \cdot \textcircled{1} \\ \textcircled{3} \leftarrow \textcircled{3} - 1 \cdot \textcircled{1}}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, we see that

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

are two linearly independent eigenvectors for $\lambda = 1$.

Altogether, we have three linearly independent eigenvectors. Hence,

$$\boxed{A \text{ is diagonalizable}}.$$

4c: Choose the eigenvectors as the columns of

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and let} \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\det(X) = 1$, the matrix X is nonsingular. Moreover,

$$X^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

and $AX = XD$. Therefore,

$$\boxed{A = XDX^{-1}}.$$

4d: For every positive integer n,

$$D^n = \begin{bmatrix} 0^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 1^n \end{bmatrix} = D.$$

Consequently,

$$A^n = XD^nX^{-1} = XDX^{-1} = A.$$

Thus,

$$\boxed{A^n = A \qquad \text{for every } n \geqslant 1}.$$

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10 pts free